Theoretical progress and status on pion scattering length

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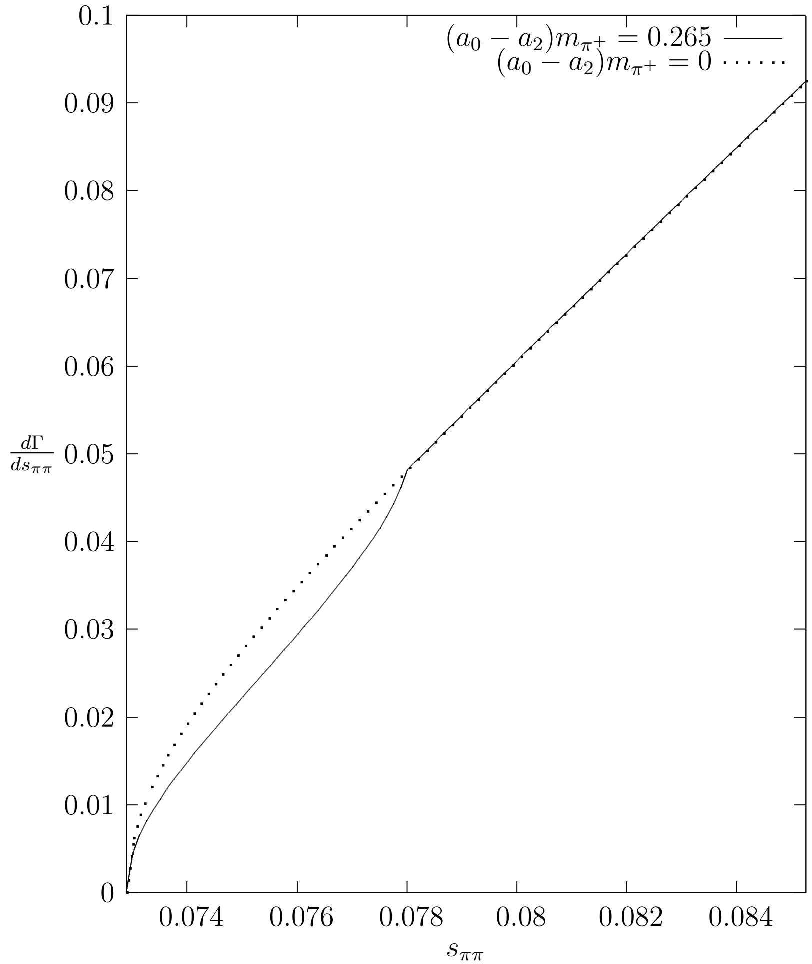
Summary

- Pion scattering lengths and the cusp.
- Unitarity and analiticity.
- $\pi\pi$ scattering.
- $K \to 3\pi$ amplitudes at $O(a_i^2)$.

N. Cabibbo, *Phys. Rev. Lett.*, 93:121801, 2004.

N. Cabibbo and G.Isidori, J. High Energy Phys. JHEP03(2005)021

How the Cusp looks



The $\pi^0\pi^0$ spectrum in $K^+ \to \pi^+\pi^0\pi^0$ in the $\pi^+\pi^-$ threshold region, at the first order in an expansion in powers of a_i

The cusp singularity is proportional to $a_0 - a_2$

Pions and QCD

The QCD Lagrangian	$\mathscr{L} = \left(\bar{\psi}_L\, i\gamma{\cdot}\mathscr{D}\, \psi_L\right) + \left(\bar{\psi}_R\, i\gamma{\cdot}\mathscr{D}\, \psi_R\right)$
(mass terms)	$+ \left(\bar{\psi}_R M\, \psi_L\right) + \left(\bar{\psi}_L M\, \psi_R\right)$
(e.m. interactions)	$+\, e\left(\bar{\psi}_L Q\, \gamma{\cdot}A\, \psi_L\right) + e\left(\bar{\psi}_R Q\, \gamma{\cdot}A\, \psi_R\right)$

M is the mass matrix and Q the charge matrix. Neglecting strange and heavier quarks,

$$M = \begin{vmatrix} m_u & 0 \\ 0 & m_d \end{vmatrix}, \qquad Q = \begin{vmatrix} 2/3 & 0 \\ 0 & -1/3 \end{vmatrix}$$

In the limit $M = 0$, $e = 0$ there is exact $SU(2)\times SU(2)$ chiral symmetry — separate isospin for left handed and right handed quarks. In the real world this symmetry is broken to $SU(2)$ — isospin, with pions acting as Nambu-Goldstone bosons. This leads to accurate predictions for the low energy pion interactions, both in π-Nucleon and in $\pi - \pi$ scattering. In comparing with experimental data also take into account isospin breaking from $m_d - m_u$ and radiative corrections.

These considerations can be extended to include the strange quark – and kaons. We will not discuss these extensions here.

Pion scattering lengths and chiral dynamics

In the limit of exact I-spin (neglect electromagnetic interactions and the $\pi^+ - \pi^0$ mass difference) we have accurate predictions for the $\pi - \pi$ scattering lenghts.

The scattering lenght defines the the low energy limit of the scattering amplitude.

Weinberg 1966 :

$$a_0 m_{\pi^+} = \frac{7 m_{\pi^+}^2}{16\pi f_\pi^2} = 0.159$$

$$a_2 m_{\pi^+} = \frac{-m_{\pi^+}^2}{8\pi f_\pi^2} = -0.045$$

Colangelo et al. 2001 :

$$a_0 m_{\pi^+} = 0.220 \pm 0.005$$

$$a_2 m_{\pi^+} = -0.0444 \pm 0.0010$$

$$(a_0 - a_2) m_{\pi^+} = 0.265 \pm 0.004$$

Can we match this precision?

The problem of I-spin breaking — a pragmatic approach

The scattering length defines the scattering amplitude at threshold for S-wave states:

$$\mathcal{M}_{II} \approx \frac{8 a_I m_\pi}{\pi} \qquad (I = 0, 2)$$

We cannot measure directly a_0 and a_2, which are defined in the unphysical limit of exact I-spin, but only the amplitutes for specific $\pi\pi$ scattering channels at the respective thresholds. Identifying m_π with m_{π^+} we will thus *define* :

$\pi^0\pi^0 \to \pi^0\pi^0$	$\mathcal{M}_{00} = \frac{8 a_{00} m_{\pi^+}}{\pi}$	$(\pi^+\pi^-$ threshold)	$a_{00} \xrightarrow{\text{I-spin}} \frac{a_0 + 2a_2}{3}$
$\pi^+\pi^0 \to \pi^+\pi^0$	$\mathcal{M}_{+0} = \frac{8 a_{+0} m_{\pi^+}}{\pi}$	$(\pi^+\pi^0$ threshold)	$a_{+0} \xrightarrow{\text{I-spin}} \frac{a_2}{2}$
$\pi^+\pi^- \to \pi^0\pi^0$	$\mathcal{M}_{x} = \frac{8 a_{x} m_{\pi^+}}{\pi}$	$(\pi^+\pi^-$ threshold	$a_{x} \xrightarrow{\text{I-spin}} \frac{a_0 - a_2}{3}$
$\pi^+\pi^- \to \pi^+\pi^-$	$\mathcal{M}_{+-} = \frac{8 a_{+-} m_{\pi^+}}{\pi}$	$(\pi^+\pi^-$ threshold)	$a_{+-} \xrightarrow{\text{I-spin}} \frac{2a_0 + a_2}{6}$
$\pi^+\pi^+ \to \pi^+\pi^+$	$\mathcal{M}_{++} = \frac{8 a_{++} m_{\pi^+}}{\pi}$	$(\pi^+\pi^+$ threshold)	$a_{++} \xrightarrow{\text{I-spin}} a_2$

I-spin breaking in $\pi\pi$ threshold scattering

In order to compare with experimental measurements of $\pi\pi$ scattering lenghts chiral perturbation theory should include the effects of I-spin breaking. A first step (private communication by G. Colangelo, J. Gasser, B. Kubis and A. Rusetsky, to be considered preliminary) was the calculation of $\mathrm{O}(e^2)$ corrections at the lowest order in Chiral perturbation theory. The correction are small but significant, e.g.

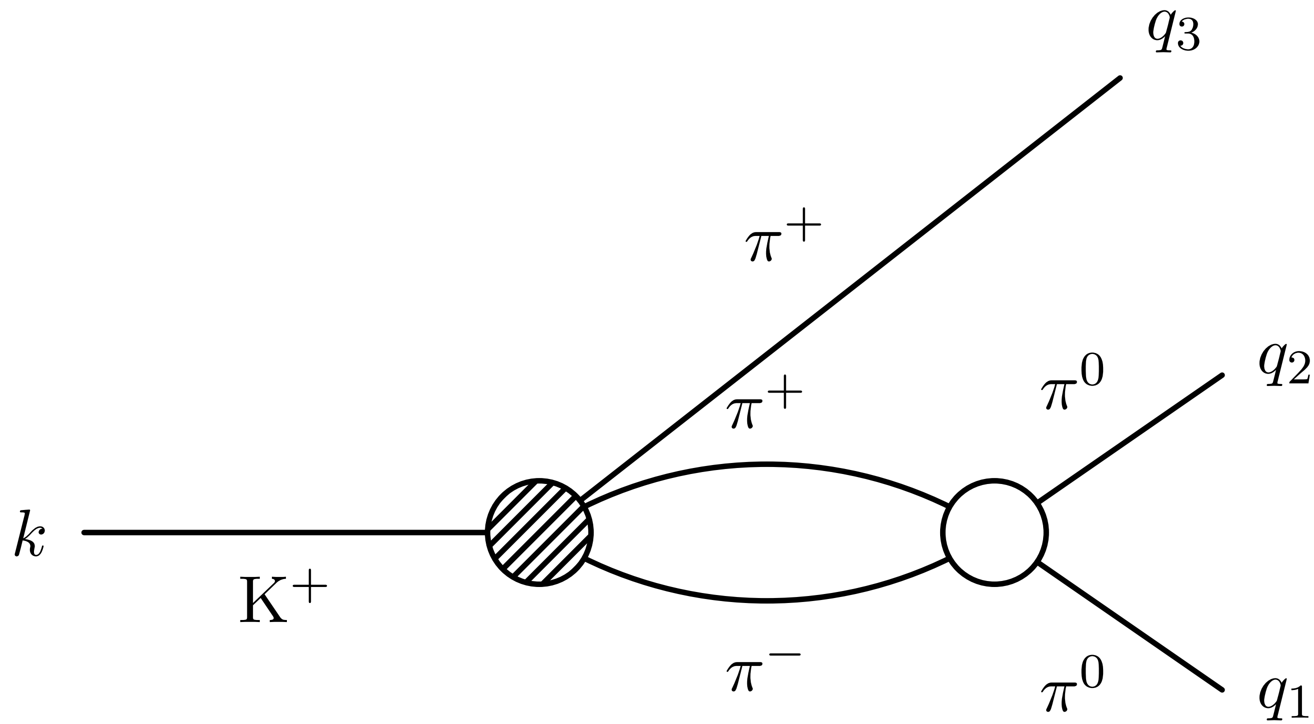
$$a_x = \frac{a_0 - a_2}{3}\left(1 + 2.2 \times 10^{-2}\right)$$

so that the chiral perturbation theory predictions for a_x should become

$$a_x m_{\pi^+} = \frac{(0.271 \pm 0.004)}{3}$$

This is not the radiative correction, but only the chiral perturbation theory correction due to the electromagnetic $m_{\pi^+} - m_{\pi^0}$ mass difference

The Cusp in $K^+ \to \pi^+\pi^0\pi^0$



$\pi^+\pi^- \to \pi^0\pi^0$ re-scattering from $K^+ \to \pi^+\pi^+\pi^-$ causes a cusp singularity in the $\pi^0\pi^0$ spectrum at the $\pi^+\pi^-$ threshold.
The cusp is proportional to $(a_0 - a_2)$

N. Cabibbo, Phys. Rev. Lett. **93** (2004) 121801.

How does the cusp arise

Let us write:

$$\mathcal{M}(K^+ \to \pi^+\pi^0\pi^0) = \mathcal{M} = \mathcal{M}_0 + \mathcal{M}_1$$

where $\mathcal{M}_1$ is the contribution of the re-scattering graph.

We must consider two cases, with $s_{\pi\pi}$ above or below the $\pi^+\pi^-$ threshold.

$$s_{\pi\pi} > 4m_{\pi^+}^2 \ : \qquad \mathcal{M}_1 = i2a_x m_{\pi^+}\mathcal{M}_{+,\mathrm{thr}}\sqrt{(s_{\pi\pi} - 4m_{\pi^+}^2)/s_{\pi\pi}}$$

$$|\mathcal{M}|^2 = (\mathcal{M}_0)^2 + |\mathcal{M}_1|^2$$

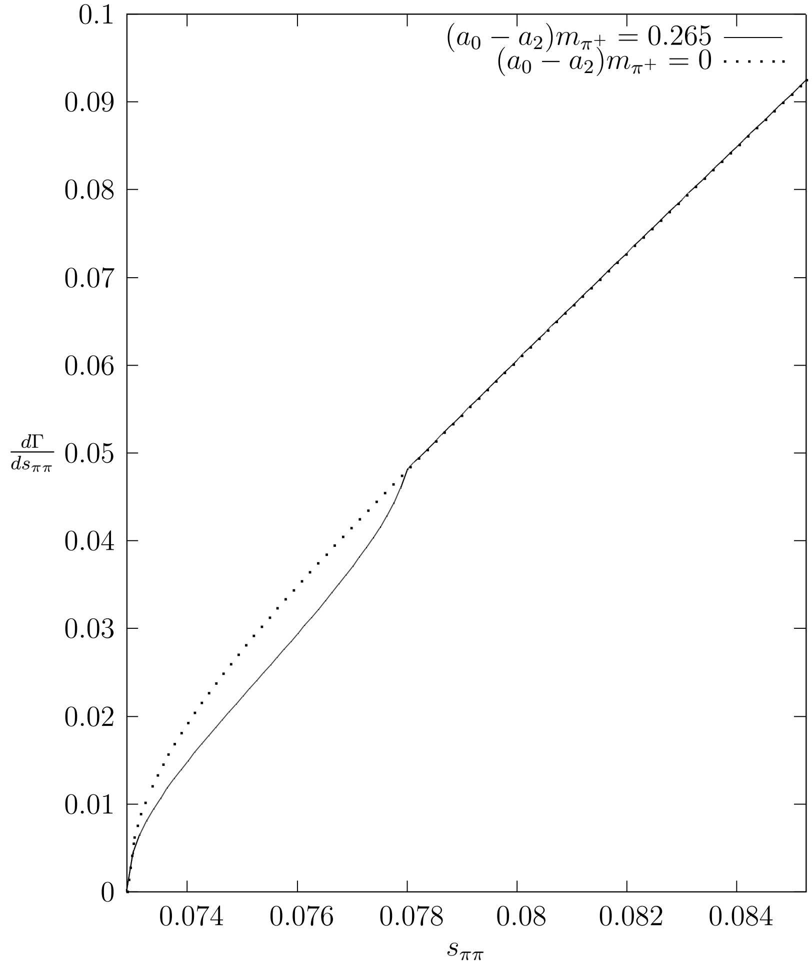
$$s_{\pi\pi} < 4m_{\pi^+}^2 \ : \qquad \mathcal{M}_1 = -2a_x m_{\pi^+}\mathcal{M}_{+,\mathrm{thr}}\sqrt{(4m_{\pi^+}^2 - s_{\pi\pi})/s_{\pi\pi}}$$

$$|\mathcal{M}|^2 = (\mathcal{M}_0)^2 + (\mathcal{M}_1)^2 + 2\mathcal{M}_0\mathcal{M}_1$$

where $\mathcal{M}_{+,\mathrm{thr}}$ is the value of the $K^+ \to \pi^+\pi^+\pi^-$ amplitude at the $\pi^+\pi^-$ threshold.

- Above threshold: $\mathcal{M}_1$ imaginary — No interference
- Below threshold: $\mathcal{M}_1$ real and negative — Interferes destructively with $\mathcal{M}_0$

How the Cusp looks



The $\pi^0\pi^0$ spectrum in $K^+ \to \pi^+\pi^0\pi^0$ in the $\pi^+\pi^-$ threshold region, at the first order in an expansion in powers of a_i

NA48 data and the cusp

With 10^8 $K^+ \to \pi^+\pi^0\pi^0$ events, a 1÷2 % measurement of $(a_0 - a_2)$ seems possible, but since the cusp is a 10% effect, this requires a theory good to $1 \div 2$ parts in 10^{-3}, and implicates higher order rescattering effects, and radiative corrections .

It is possible to set up a systematic computation of the singular parts of an amplitude in terms of its non-singular parts. This leads to an expansion of the $K \to 3\pi$ amplitudes in powers of the $\pi\pi$ scattering lengths a_0, a_2.

$3\pi \to 3\pi$ scattering amplitudes are also implicated but give negligible correction at our target precision

The development is useful because the scattering lengths are small, which is a general consequence of the fact that pions act as "pseudo Goldstone Bosons" for chiral symmetry breaking.

The development to the second order in powers of a_0, a_2 reveals a second cusp above the $\pi^+\pi^-$ threshold.

New cusp results

With Gino Isidori we have completed a computation of the $O(a_i^2)$ corrections to the $K \to 3\pi$ amplitudes. This work includes:

- Other rescattering corrections: $\pi^0\pi^0 \to \pi^0\pi^0$, $\pi^+\pi^0 \to \pi^+\pi^0$, etc.
- Corrections from “two-loops” graphs.
- It covers $K^+ \to \pi^+\pi^0\pi^0$, $K_L \to \pi^0\pi^0\pi^0$, which have cusps, and $K^+ \to \pi^+\pi^+\pi^-$, $K_L \to \pi^+\pi^-\pi^0$. which do not.

This work allows for a determination of $a_0 - a_2$ with 5% theoretical uncertainty. To reach a 1% theoretical uncertainty we need:

- An evaluation of smaller (three loop) corrections.
- An evaluation of radiative corrections.

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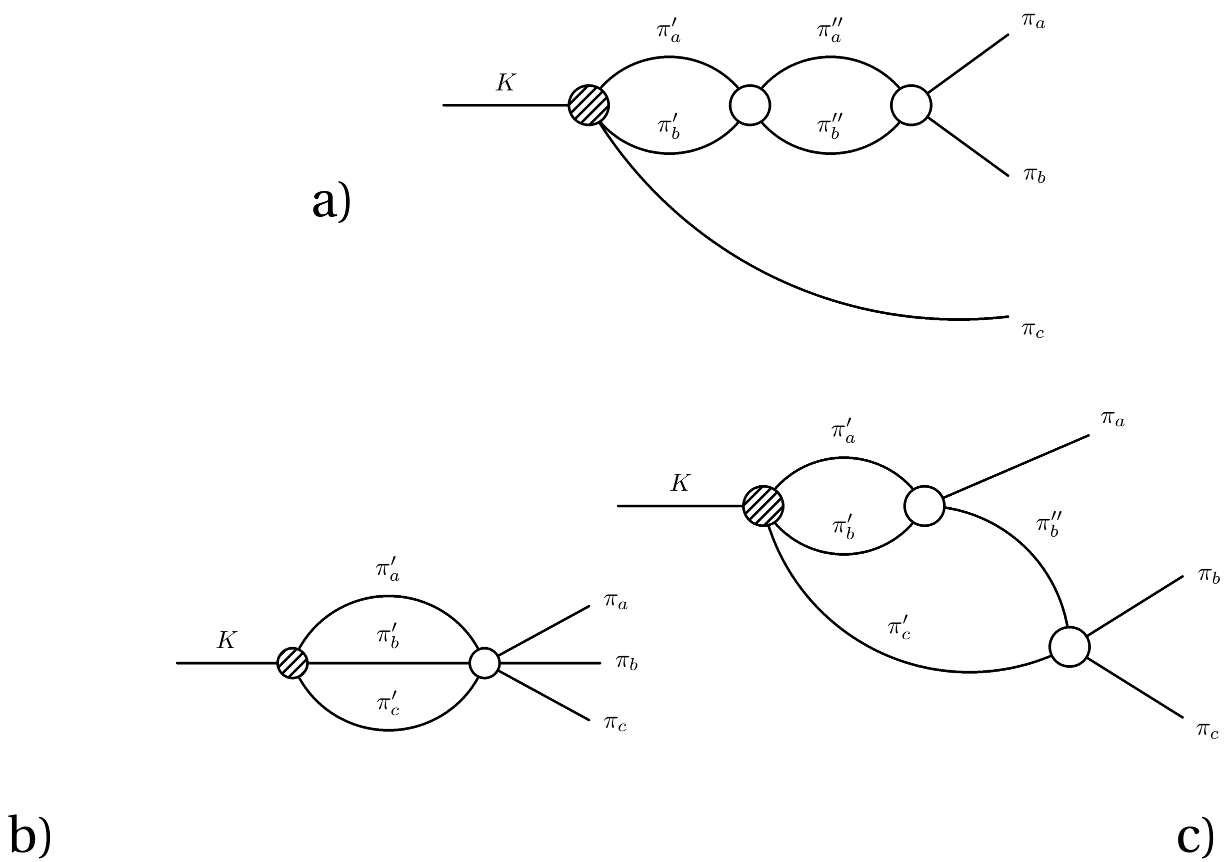


$K \to 3\pi$ rescattering topologies at the two-loop level:

a) single-channel $\pi\pi$ scattering;

b) irreducible $3\pi \to 3\pi$ contributions;

c) $3\pi \to 3\pi$ amplitude due to $\pi\pi$ scattering.

Time reversal, Unitarity and Analyticity

The structure of the cusp is largely determined by very general principles.

1. Time reversal: In the sectors of interest the **S**-matrix is symmetric,

$$\langle B|\mathbf{S}|A\rangle = \langle A|\mathbf{S}|B\rangle \qquad K \to 3\pi \text{ and } \pi\pi \to \pi\pi$$

2. Unitarity:

$$\mathbf{S} = \mathbf{1} + i(\mathbf{R} + i\mathbf{I}) \qquad \text{where } \mathbf{R} \text{ and } \mathbf{I} \text{ are hermitian.}$$

   From time reversal: **R** and **I** are symmetric, their matrix elements are the real and imaginary parts of the matrix elements of **S**.

$$2\mathbf{I} = \mathbf{R}^2 + \mathbf{I}^2 \qquad \text{or, solving for } \mathbf{I},$$

$$\mathbf{I} = \mathbf{1} - \sqrt{\mathbf{1} - \mathbf{R}^2} \qquad = \qquad \frac{1}{2}\mathbf{R}^2 + \frac{1}{8}\mathbf{R}^4 + \frac{1}{16}\mathbf{R}^6 + \frac{5}{128}\mathbf{R}^8 \ldots$$

3. Analyticity.

$\pi^0\pi^0 \to \pi^0\pi^0$ scattering at $\mathbf{O}(a_i^3)$ — 1

Define the "velocities",

$$v_\pm(s) = \sqrt{\frac{|s - 4m_{\pi^+}^2|}{s}} \qquad v_{00}(s) = \sqrt{\frac{|s - 4m_{\pi^0}^2|}{s}}$$

We can then write

$$\mathcal{M}_{00} = A_{00} + B_{00}\, v_\pm(s) \qquad s > 4m_{\pi^+}^2$$

$$\mathcal{M}_{00} = A_{00} + i B_{00}\, v_\pm(s) \qquad s < 4m_{\pi^+}^2,$$

and, for $\pi^+\pi^- \to \pi^0\pi^0$,

$$\mathcal{M}_x = A_x + B_x\, v_\pm(s) \qquad s > 4m_{\pi^+}^2$$

where A_{00}, B_{00}, A_x, B_x are regular (analytic) at the $\pi^+\pi^-$ threshold.

$\pi^0\pi^0 \to \pi^0\pi^0$ scattering at $O(a_i^3)$ — 2

We can express Re(A) as a polynomial in s. Write:

$$\mathrm{Re}(A_{00}) = \frac{8a_{00}(s)}{\pi}; \qquad a_{00}(s) = a_{00}\left[1 + r_{00}\frac{(s-4m_{\pi^+}^2)}{4m_{\pi^+}^2} + \dots\right]$$

and similarly for $\pi^+\pi^- \to \pi^0\pi^0$,

$$\mathrm{Re}(A_x) = \frac{8a_x(s)}{\pi}; \qquad a_x(s) = a_x\left[1 + r_x\frac{(s-4m_{\pi^+}^2)}{4m_{\pi^+}^2} + \dots\right]$$

In the limit of exact SU(2), $a_{00} = \frac{a_0+2a_2}{3}$, $a_x = \frac{a_0-a_2}{3}$

The $\pi^+\pi^-$ intermediate state contributes to Im$\mathcal{M}_{00}$ only above the $\pi^+\pi^-$ threshold, while the $\pi^0\pi^0$ state contributes both above and below, so that, at $O(\mathbf{R}^2)$,

$$\mathrm{Im}\,\mathcal{M}_{00} = \frac{\pi}{4}v_\pm(s)\,(\mathrm{Re}\,\mathcal{M}_x)^2\,\Theta(s-4m_{\pi^+}^2) + \frac{\pi}{8}v_{00}(s)\,(\mathrm{Re}\,\mathcal{M}_{00})^2 + O(\mathbf{R}^4)$$

$\pi^0\pi^0 \to \pi^0\pi^0$ scattering at $O(a_i^3)$ — 3

Note that $\quad \mathrm{Im}\,\mathcal{M}_{00} = \begin{cases} \mathrm{Im}\,A_{00} + \mathrm{Im}\,B_{00}\, v_\pm(s) & : \quad s > 4m_{\pi^+}^2 \\ \mathrm{Im}\,A_{00} + \mathrm{Re}\,B_{00}\, v_\pm(s) & : \quad s < 4m_{\pi^+}^2 \end{cases}$

Applying unitarity, $\mathbf{I} = \mathbf{R}^2/2$ at $O(\mathbf{R}^2)$ both above and below the $\pi^+\pi^-$ threshold we then obtain

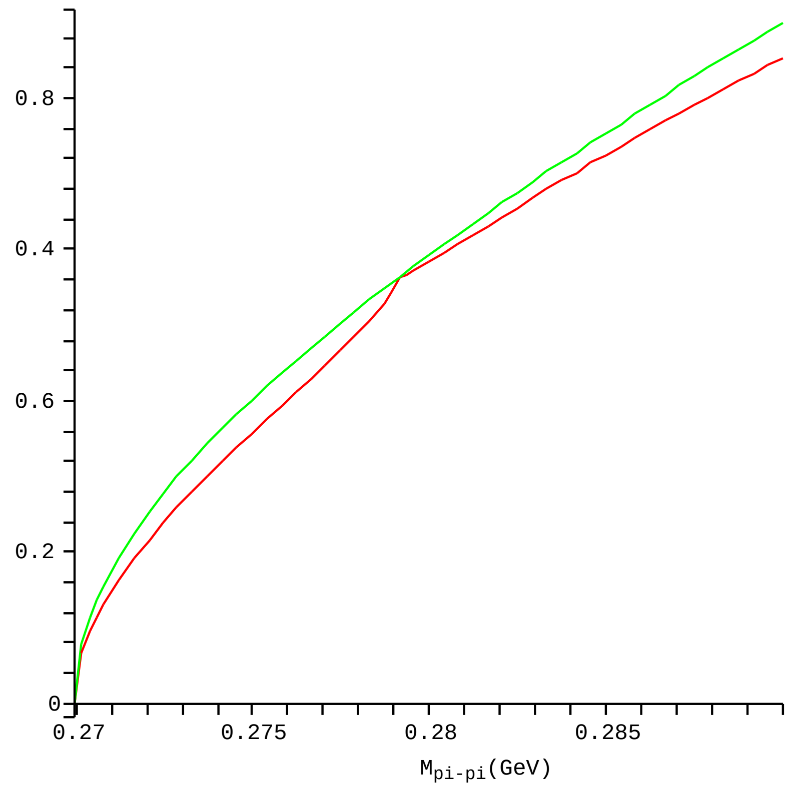
$$\mathrm{Im}\,B_{00} = \frac{\pi}{4}(\mathrm{Re}\,A_x)^2 = \frac{16}{\pi}\,(a_x(s))^2$$

$$\mathrm{Re}\,B_{00} = -\frac{\pi\, v_{00}(s)}{4}\,\mathrm{Re}\,A_{00}\,\mathrm{Im}\,B_{00} = -\frac{32\, v_{00}(s)}{\pi}\, a_{00}(s)(a_x(s))^2$$

$$\mathrm{Im}\,A_{00} = \frac{\pi\, v_{00}(s)}{8}(\mathrm{Re}\,A_{00})^2 = \frac{8\, v_{00}(s)}{\pi}\,(a_{00}(s))^2$$

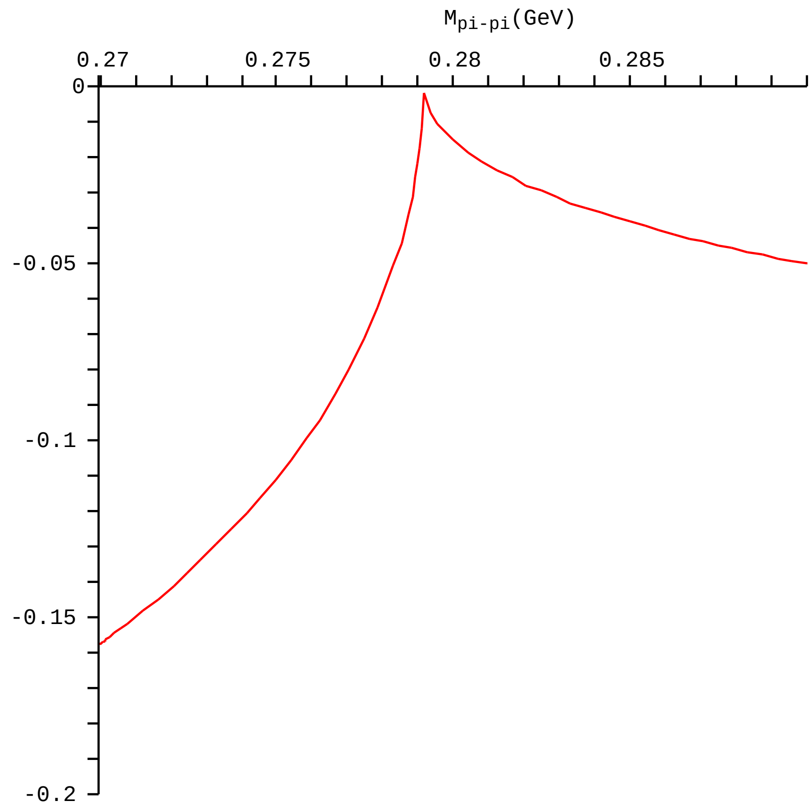
These results (and similar ones for other channels) are then fed into the unitarity relations for $K \to 3\pi$ to obtain all corrections at $O(a_i^2)$.

The cusp at two loops



The theoretical $\pi^0\pi^0$ spectrum in $K^+ \to \pi^+\pi^0\pi^0$ (lower) computed at second order in powers of the scattering lengths. At the two-loops level also the “non-cusp” amplitude has an imaginary part arising from rescattering in $\pi^+\pi^0 \to \pi^+\pi^0$, etc.

The cusp region



Theoretical $\pi^0\pi^0$ spectrum in $K^+ \to \pi^+\pi^0\pi^0$ (lower) at $O(a_i{}^2)$ normalized to the "unperturbed" amplitude.

The picture will be complicated by the existence of a $\pi^+\pi^-$ pionium atom signal at the $\pi^+\pi^-$ threshold!.

Space for improvement

- Stopping at $O(a_i^2)$ level and neglecting radiative corrections implies a ~ 5% theoretical error on $(a_0 - a_2)$. This may be sufficent at the present state of NA48 systematics.
- $O(a_i^3)$ terms and radiative corrections can be computed with a finite, but lengthy effort.
- Compute $O(a_i^3)$ terms.
- Compute radiative corrections
- These computations will to a large extent be independent from Chiral Perturbation Theory which we wish to test!

Conclusions

- The experimental study of $K \to 3\pi$ decays is a powerful tool for gathering information on $\pi\pi$ scattering in the low energy region.
- One can systematically evaluate rescattering effects in $K \to 3\pi$ decays by means of an expansion in powers of the $\pi\pi$ scattering lengths, and of other $\pi\pi$ scattering parameters.
- This approach is less ambitious than the ordinary loop expansion performed in effective field theories, such as CHPT: the scope is not a dynamical calculation of the entire decay amplitudes, but a systematic evaluation of the singular terms due to rescattering effects.
- We have explicitly computed all the $O(a_i^2)$ corrections to the leading cusp effect in $K^+ \to \pi^+\pi^0\pi^0$ The new terms produce a smaller square-root cusp behavior also above the $\pi^+\pi^-$ singularity.
- The 5% level of precision is probably not sufficient to fully exploit the potentially very accurate data of NA48, and is also larger than the error on the CHPT predictions of $a_0 - a_2$. To improve the situation we need a complete evaluation of the $O(a_i^3)$ corrections and of the effects due to radiative corrections. Only first steps in this direction have been taken
- The theoretical error on $(a_0 - a_2)$ can be reduced to the 1% level at both in theory and experiment. This might turn into the most accurate test of Chiral Perturbation Theory