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Bern-Bonn collaboration

Cusps in the kaon decays

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PLB 638 (2006) 187, PLB 659 (2008) 576,
work in preparation

Meson 2008, Kraków, 6-10 June 2008

Plan

- $\pi\pi$ scattering lengths
- Cusps in the $K \to 3\pi$ decays

    Experiment

    Physics background

- Non-relativistic effective theory

    Essentials

    $K \to 3\pi$ *amplitudes at 2 loops*

    Including photons

- $\pi\pi$ scattering lengths from K_{e_4} decays

    Analysis of the experimental data

    Fate of Watson's theorem in case of isospin breaking

    Corrections due to $m_d - m_u$

- Conclusions

The $\pi\pi$ scattering lengths

Two-loop Chiral perturbation theory + Roy equations:

G. Colangelo, J. Gasser and H. Leutwyler, PLB 488 (2000) 261; NPB 603 (2001) 125

$$a_0 = 0.220 \pm 0.005\,, \qquad a_2 = -0.0444 \pm 0.0010$$

- *Theoretical precision* $\simeq 1.5\%$
- *Test of large/small condensate scenario in QCD*

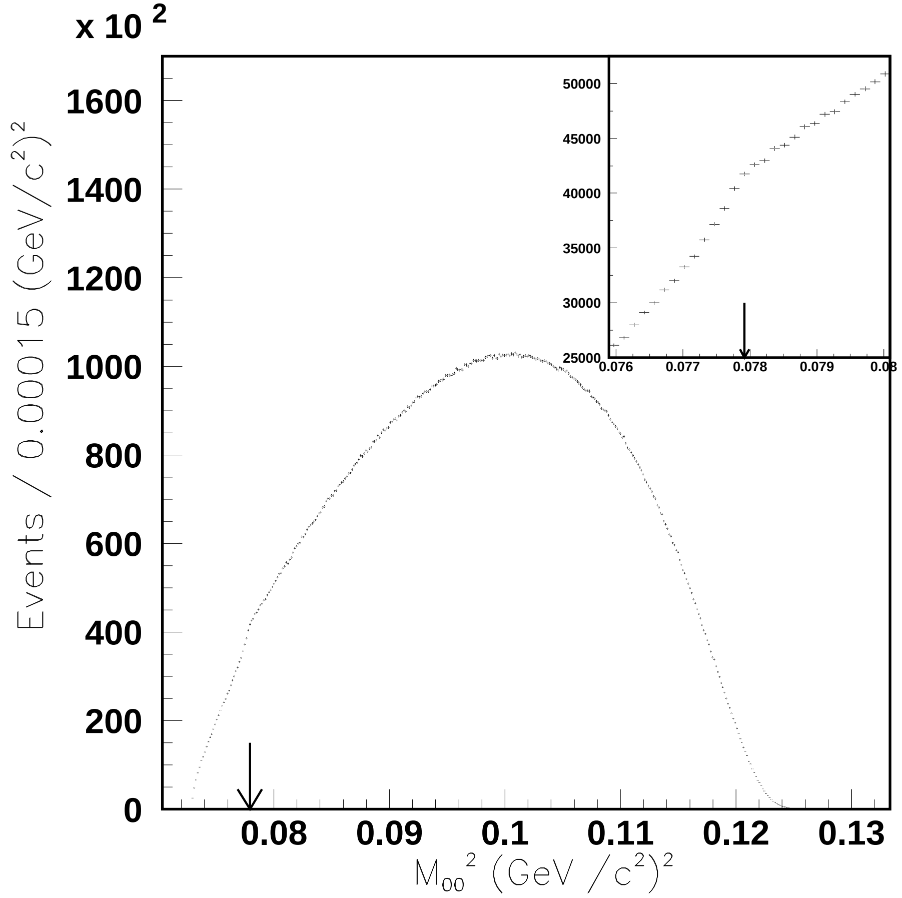
Experiments to measure $\boldsymbol{\pi\pi}$ scattering lengths:

- *Cusps in $K \to 3\pi$ decays: NA48/2 (CERN)*
- *K_{e_4} decays: Geneva-Saclay, E865 (BNL), NA48/2 (CERN)*
- *Pionium lifetime: DIRAC (CERN)*
- *$\pi N \to \pi\pi N$: Berkeley, CERN-Munich, Kurchatov Inst.*

Cusps

- $M_\pi \neq M_{\pi^0} \quad \Rightarrow \quad$ *cusps in the decay amplitudes*
- $K \to 3\pi$ decays:
  - *The cusp is kinematically accessible*
  - a_0, a_2 *can be extracted from measuring the parameters of the cusp*
- $K^+ \to \pi^+\pi^- e^+\nu_e$:
  - *The cusp in the phase at* $\pi^+\pi^-$ *threshold*
  - *Modifies the analytic structure of the amplitude in the vicinity of threshold*
  - *Comparing experiment with theory, isospin-breaking corrections should be taken into account*

The cusp in the $\pi^0\pi^0$ invariant mass distribution (NA48/2)



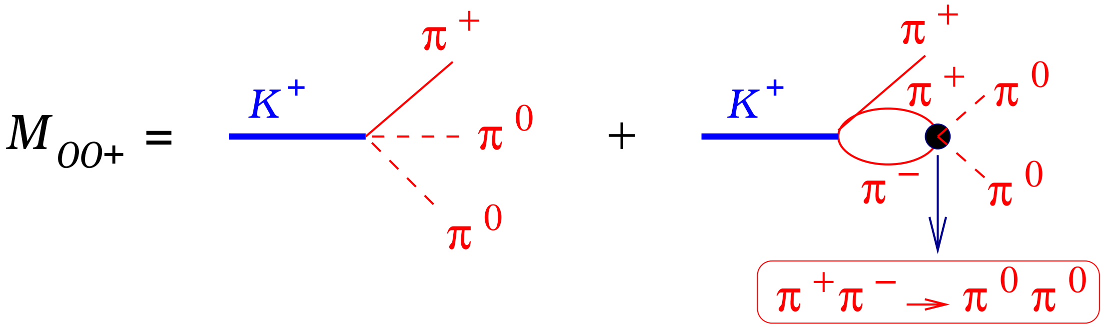
$K^+ \to \pi^+\pi^0\pi^0$

Partial sample of $\sim 2.3 \cdot 10^7$ decays

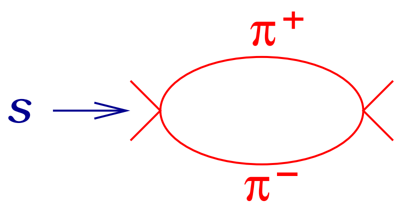
J. R. Batley *et al.* [NA48/2 Collaboration], PLB 633 (2006) 173

Heuristic theory of the cusp

Interference of tree + 1 loop *(N. Cabibbo, PRL 93 (2004) 121801)*



$$\boxed{\frac{d\Gamma_{00+}}{ds_{\pi\pi}} \propto \int |\mathcal{M}_{00+}|^2}$$



$$= \text{“smooth”} + \frac{i}{16\pi} \underbrace{\left(1 - \frac{4M_\pi^2}{s}\right)^{1/2}}_{=\,\sigma_c(s)}$$

Parameters of the cusp $\Rightarrow$ S-wave $\pi\pi$ scattering lengths a_0, a_2

At present experimental precision a simple parameterization of the cusp does not suffice. A systematic theoretical framework is needed that describes $K \to 3\pi$ in the vicinity of the cusp

$K \to 3\pi$ decays: theory

- N. Cabibbo and G. Isidori, JHEP 0503 (2005) 021:

  Parameterization of the decay amplitudes up to and including two loops, using analyticity and unitarity

- Also: E. Gamiz, J. Prades, and I. Scimemi, EPJC 50 (2007) 405:

  A supplementary approach, merger to ChPT at one loop

$\Rightarrow$ *Not a full dynamical scheme (photons?)*

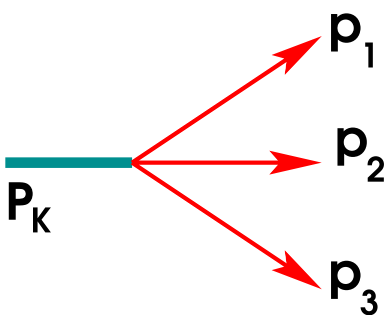
$\Rightarrow$ *The analytic ansatz for the amplitudes, which has been assumed, is not valid beyond one loop*

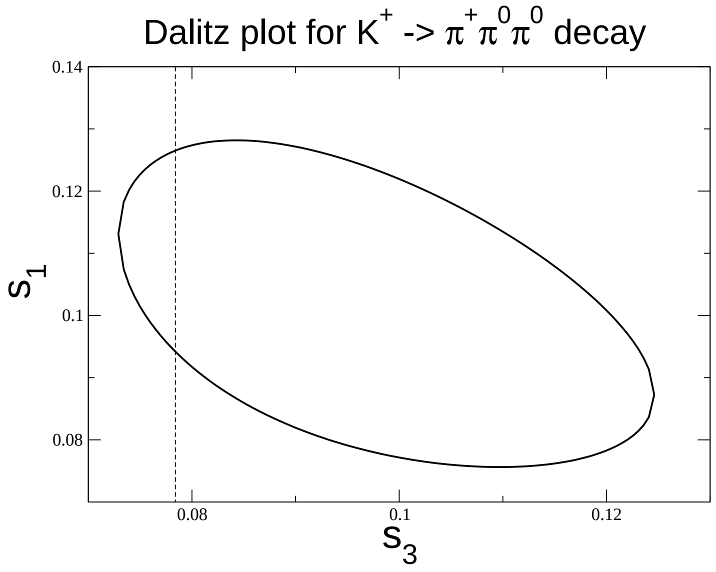
One needs a systematic theory of $K \to 3\pi$, which would provide a reliable control on the accuracy!

$K \to 3\pi$ decays: the kinematics

Neutral mode : $K^+ \to \pi^0\pi^0\pi^+$

Charged mode : $K^+ \to \pi^+\pi^+\pi^-$





$$s_i = (P_K - p_i)^2 \qquad p_i^2 = M_i^2\,, \quad i = 1,2,3$$

Non-relativistic description is valid in the region:

$|\mathbf{p}_i|/M_i = O(\epsilon)$, *small momenta*

$T_i = w(\mathbf{p}_i) - M_i = O(\epsilon^2)$, *small kinetic energies*

$$M_K - \sum_i M_i = \sum_i T_i = O(\epsilon^2) \ll M_i, \quad \Delta M_\pi^2 = O(\epsilon^2)$$

Non-relativistic EFT: essentials

Bern-Bonn coll., PLB 638 (2006) 187, PLB 659 (2008) 576

⇒ *Include distant singularities, emerging in relativistic QFT, into the effective couplings of non-relativistic Lagrangian*

$$\frac{1}{M_\pi^2 - p^2} = \underbrace{\frac{1}{2w(\mathbf{p})}\,\frac{1}{w(\mathbf{p}) - p^0}}_{\text{particles}} + \underbrace{\frac{1}{2w(\mathbf{p})}\,\frac{1}{w(\mathbf{p}) + p^0}}_{\text{antiparticles}}$$

$$w(\mathbf{p}) = M_\pi + \frac{\mathbf{p}^2}{2M_\pi} - \frac{\mathbf{p}^4}{8M_\pi^3} + \cdots \qquad \text{at} \quad |\mathbf{p}| \ll M_\pi$$

- *Two-particle sector: Lagrangian*

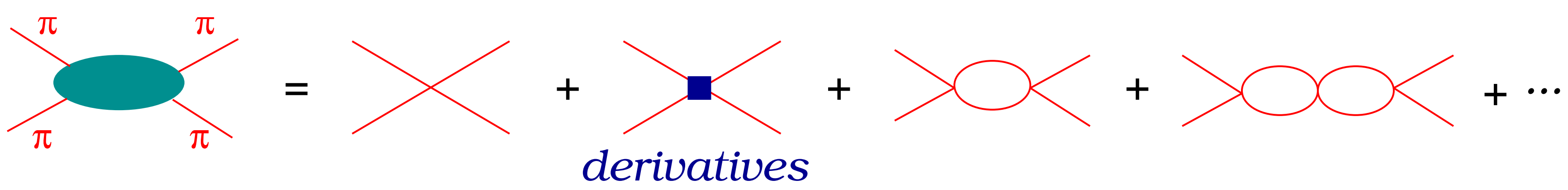
$$\mathcal{L}_{NR} = \Phi^\dagger (2W)(i\partial_t - W)\Phi + C_0 \Phi^\dagger \Phi^\dagger \Phi \Phi + \text{deriv. couplings}$$

⇒ *Do loops with this Lagrangian in dim. reg. + threshold expansion*

⇒ *Covariant 2-particle scattering amplitude in moving frames*

Why non-relativistic theory?

- *It is a full dynamical scheme based on a Lagrangian (photons!)*
- *Analyticity + unitarity automatically taken into account*



$\Rightarrow$ *Each loop $\propto i|\mathbf{p}|$, vanishes at threshold, $O(\epsilon)$ suppressed*

$$\begin{array}{lccccccc} \mathrm{Re}\,T_{NR} & = & a & + & b\mathbf{p}^2 & + & c\mathbf{p}^4 & + \cdots \\ & & \text{tree} & & \text{tree} + \text{twoloop} & & \cdots & \end{array}$$

Non-relativistic theory = effective range expansion

Non-relativistic couplings = scattering lengths $a, \cdots$

On the contrary, in ChPT: $a = O(M_\pi^2) + O(M_\pi^4) + O(M_\pi^6) + \cdots$

Non-relativistic approach for $K \longrightarrow 3\pi$ decays

$$\mathcal{L}_{\pi\pi} = C_x(\pi_-^\dagger \pi_+^\dagger \pi_0 \pi_0 + \text{h.c.}) + \cdots, \qquad C_x = (a_0 - a_2) + \text{isospin br.}$$

$$\mathcal{L}_{K^+ \to \pi^0 \pi^0 \pi^+} = \frac{G_0}{2}(K^\dagger \pi_+ \pi_0^2 + \text{h.c.}) + \cdots$$

$$\mathcal{L}_{K^+ \to \pi^+ \pi^+ \pi^-} = \frac{H_0}{2}(K^\dagger \pi_- \pi_+^2 + \text{h.c.}) + \cdots$$

$$\ldots$$

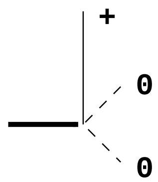
- Non-relativistic region = whole decay region, and slightly beyond
- Double expansion in:

  a (scattering lengths, effective ranges. . .) and ϵ (small momenta)

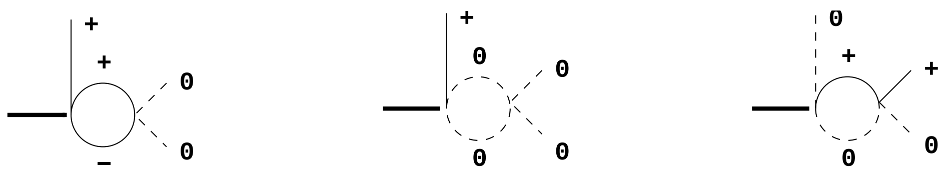
- Expansion in a and ϵ are correlated: adding one pion loop increases powers of both a and ϵ by one
- One expects that the expansion in a is convergent, as $a \ll 1$

The graphs $K^+ \to \pi^0\pi^0\pi^+$

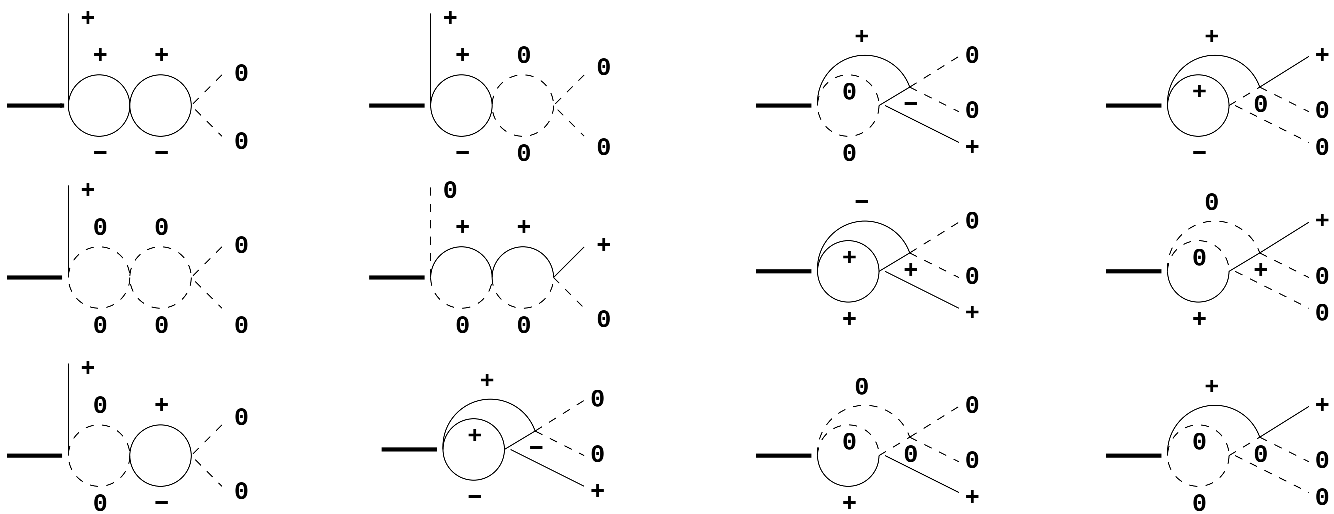
Tree:



1-loop:



2-loops:



The result for $K^+ \to \pi^0\pi^0\pi^+$

⇒ *Our result:* $O(\epsilon^4)$, $O(a\epsilon^5)$, $O(a^2\epsilon^2)$; *valid in the whole NR region*

$$\mathcal{M}_N(s_1,s_2,s_3) = \underbrace{\mathcal{M}_N^{\text{tree}} + \mathcal{M}_N^{1-\text{loop}} + \mathcal{M}_N^{2-\text{loops}} + \cdots}_{O(\epsilon^4)+O(a\epsilon^5)+O(a^2\epsilon^2)+\cdots}$$

$$\mathcal{M}_N^{\text{tree}} = G_0 + G_1(p_3^0 - M_\pi) + G_2(p_3^0 - M_\pi)^2 + G_3(p_1^0 - p_2^0)^2 + \cdots$$

$$\mathcal{M}_N^{1-\text{loop}} = B_{N1}\, J_{+-}(s_3) + B_{N2}\, J_{00}(s_3) + \left[B_{N3}\, J_{+0}(s_1) + (s_1 \leftrightarrow s_2)\right]$$

$$\mathcal{M}_N^{2-\text{loops}} = \underbrace{4H_0C_xC_{+-}J_{+-}^2(s_3) + \cdots}_{\textit{double loops}}$$

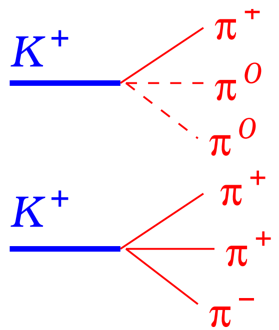
$$+ \underbrace{4H_0C_xC_{+-}F_+(M_\pi,M_\pi,M_\pi,M_\pi;s_3) + \cdots}_{\textit{overlapping loops}}$$

⇒ *Similar result for* $K^+ \to \pi^+\pi^+\pi^-$ *decay amplitude*

The strategy for determining scattering lengths

$K^+ \to \pi^0\pi^0\pi^+$ *and* $K^+ \to \pi^+\pi^+\pi^-$ *decay amplitudes depend on:*

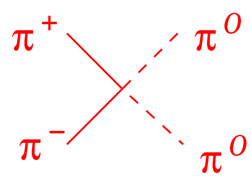
$K \to 3\pi$ decay couplings



$G_0, G_1, G_2, G_3, \ldots$

$H_0, H_1, H_2, H_3, \ldots$

$\pi\pi$ effective range parameters



C_i, D_i, E_i, F_i, ...

O(1) — O(p²) — O(p⁴)

Fit $G_i, H_i, C_i, \cdots$ *to the decay data;* $C_i \Rightarrow \pi\pi$ *scattering lengths*

$$a_0 - a_2 = 0.268 \pm 0.010\,(stat) \pm 0.004\,(syst) \pm 0.013\,(ext)$$

$$a_2 = -0.041 \pm 0.022\,(stat) \pm 0.014\,(syst)$$

B. Bloch-Devaux (NA48/2 coll.), NPB 174 (2007) 91 (proc. suppl.)

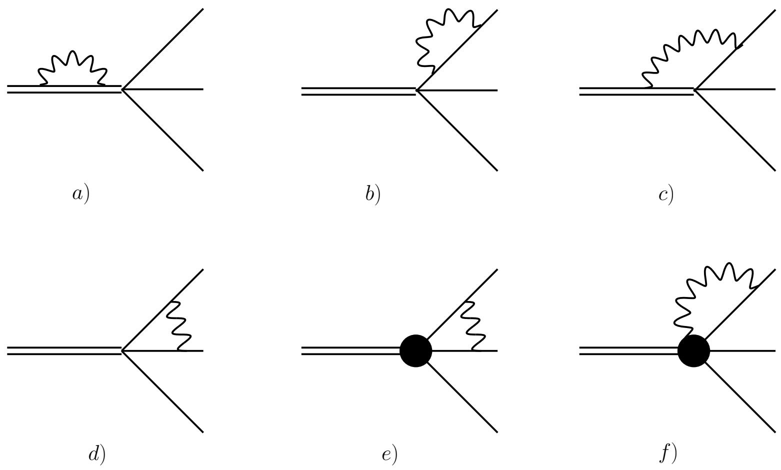
Including photons in the non-relativistic EFT

M. Bissegger, A. Fuhrer, J. Gasser, B. Kubis and AR, work in preparation

- *Minimal substitution:*

$$\partial_\mu \Phi_\pm \to (\partial_\mu \mp ieA_\mu)\Phi_\pm\,, \qquad \partial_\mu K_+ \to (\partial_\mu - ieA_\mu)K_+$$

- *Add all possible non-minimal gauge-invariant terms*



+ 2-loop diagrams

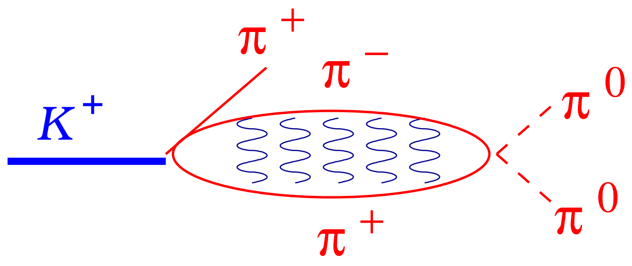
+ Bremsstrahlung

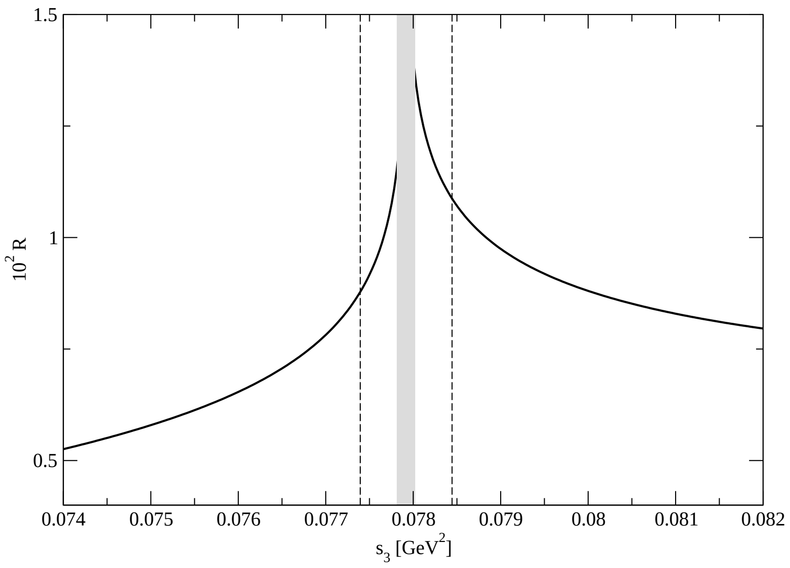
$$\left.\frac{d\Gamma}{ds_3}\right|_{E_\gamma<E_{max}} = \frac{d\Gamma(K\to 3\pi)}{ds_3} + \left.\frac{d\Gamma(K\to 3\pi\gamma)}{ds_3}\right|_{E_\gamma<E_{max}} + O(\alpha^2)$$

Coulomb photons

- *Singularity structure changed at threshold at $O(\alpha)$*
  see also S.R. Gevorkyan *et al*, hep-ph/0612129, hep-ph/0702154

$$i\sigma(s) \to i\sigma(s) - \alpha \ln \sigma(s)$$





- *Coulomb corrections are perturbative everywhere except a very small region around the cusp – exclude this region*
- *Result: a systematic parameterization of the decay amplitudes, including real and virtual photons*

The decay $K^+(p) \to \pi^+(p_1)\pi^-(p_2)e^+(p_e)\nu_e(p_\nu)$

Kinematics:

$s_\pi = (p_1+p_2)^2$, $t = (p_1-p_2)^2$, $u = (p_2-p)^2$, $s_l = (p_e+p_\nu)^2$

$$\mathcal{M} = \frac{G_F}{\sqrt{2}} V_{us}^* \langle \pi^+\pi^- | V^\mu - A^\mu | K^+ \rangle \, \bar{u}(p_\nu)\gamma_\mu(1-\gamma_5)v(p_e)$$

$$\langle \pi^+\pi^- | V^\mu - A^\mu | K^+ \rangle = \frac{-i}{M_K}\left(F(p_1+p_2)^\mu + G(p_1-p_2)^\mu\right) + \cdots$$

Partial-wave expansion:

$$F_1 = F + \frac{(M_K^2 - s_\pi - s_l)\sigma}{\lambda^{1/2}(M_K^2, s_\pi, s_l)} \cos\theta_\pi \, G \,, \quad F_1 = \sum_k P_k(\cos\theta_\pi) f_k(s_\pi, s_l)$$

Watson theorem (isospin symmetric world):

$$f_k(s_\pi + i\varepsilon, s_l) = \mathrm{e}^{2i\delta_k} f_k(s_\pi - i\varepsilon, s_l)\,, \quad \left\{ \begin{array}{l} \delta_0 = \delta_0^0 \\ \delta_1 = \delta_1^1 \end{array} \right. \Rightarrow \quad \text{measure } \delta_0^0 - \delta_1^1$$

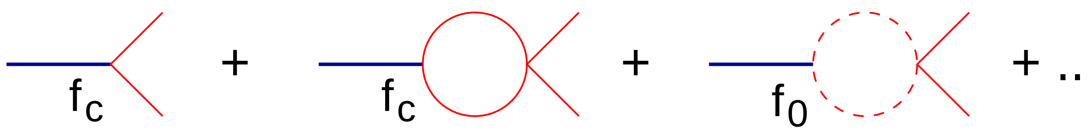
Isospin breaking (the scalar formfactor)

G. Colangelo, J. Gasser and AR, work in preparation

$$
\begin{aligned}
-F_c(s) &= \langle 0|\mathcal{O}(0)|\pi^+(p_1)\pi^-(p_2);\text{in}\rangle \\
F_0(s) &= \langle 0|\mathcal{O}(0)|\pi^0(p_1)\pi^0(p_2);\text{in}\rangle
\end{aligned}
\quad , \quad F = \begin{pmatrix} F_c \\ F_0 \end{pmatrix}
$$

Non-relativistic effective Lagrangian:

$$
\mathcal{L}_{\mathcal{O}} = \mathcal{O}\left(-f_c\Phi_+^\dagger\Phi_-^\dagger + \frac{f_0}{2}\,\Phi_0^\dagger\Phi_0^\dagger\right) + \text{h.c.} + \text{deriv. terms}
$$



Unitarity relation:

$$
\begin{aligned}
\text{Im}\,F(s) &= T(s)\rho(s)F^*(s) \\
\text{Im}\,T(s) &= T(s)\rho(s)T^*(s)
\end{aligned}
\quad ,
\quad
\begin{aligned}
T &= \begin{pmatrix} t_{cc} & -t_{c0} \\ -t_{c0} & t_{00} \end{pmatrix} \\
\rho &= \text{diag}\,(2\sigma_c, \sigma_0)
\end{aligned}
$$

Watson theorem in case of isospin breaking

Isospin symmetry limit $F_c = F_0$*:*

$$\text{Im}\, F_c = t_0^0\, \sigma_c\, F_c^* \quad \Rightarrow \quad F_c = \mathrm{e}^{i\delta_0^0}\, |F_c|$$

Isospin broken: $F = T \cdot R$ *,* $\quad R = \begin{pmatrix} R_c \\ R_0 \end{pmatrix}$ *is real,*

$\beta(s) = R_c/R_0$*, depends on* f_c/f_0 *(not fixed by* $\pi\pi$ *interaction)*

$$\tan \delta_c = \frac{\text{Im}\, t_{cc} + \beta(s)\text{Im}\, t_{c0}}{\text{Re}\, t_{cc} + \beta(s)\text{Re}\, t_{c0}}\,, \quad \tan \delta_0 = \frac{\text{Im}\, t_{c0} + \beta(s)\text{Im}\, t_{00}}{\text{Re}\, t_{c0} + \beta(s)\text{Re}\, t_{00}}$$

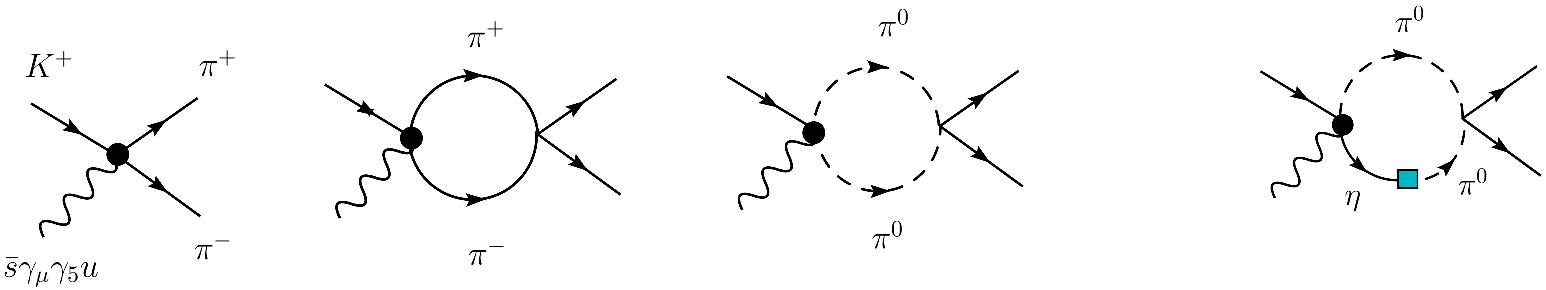
- Phases δ_c, δ_0 are not determined by the $\pi\pi$ amplitude alone
- $\text{Im}\, F_c\Big|_{s=4M_\pi^2} = -\left(1 - \frac{M_{\pi^0}^2}{M_\pi^2}\right)^{1/2} t_{c0} F_0^* \quad \Rightarrow \quad \delta_c \neq 0$ at $s = 4M_\pi^2$
- $F_c = \mathrm{e}^{i\delta_c}\hat{F}_c$ with $\hat{F}_c \propto \sigma_c \sigma_0 + \cdots$ *non-analytic* at $s = 4M_\pi^2$

cf with S.R. Gevorkyan et al, hep-ph 0704.2675, 0711.4618

One-loop result in ChPT

⇒ Use ChPT with no photons, calculate isospin-breaking corrections, subtract from the measured phase shifts

$$\mathcal{L}_2 \to \mathcal{L}_2 + C\langle QUQU^\dagger \rangle \,, \cdots \qquad Q = \frac{e}{3}\,\text{diag}\,(2,-1,-1)$$



Scalar FF: $\qquad\qquad$ $m_d - m_u$ effect:

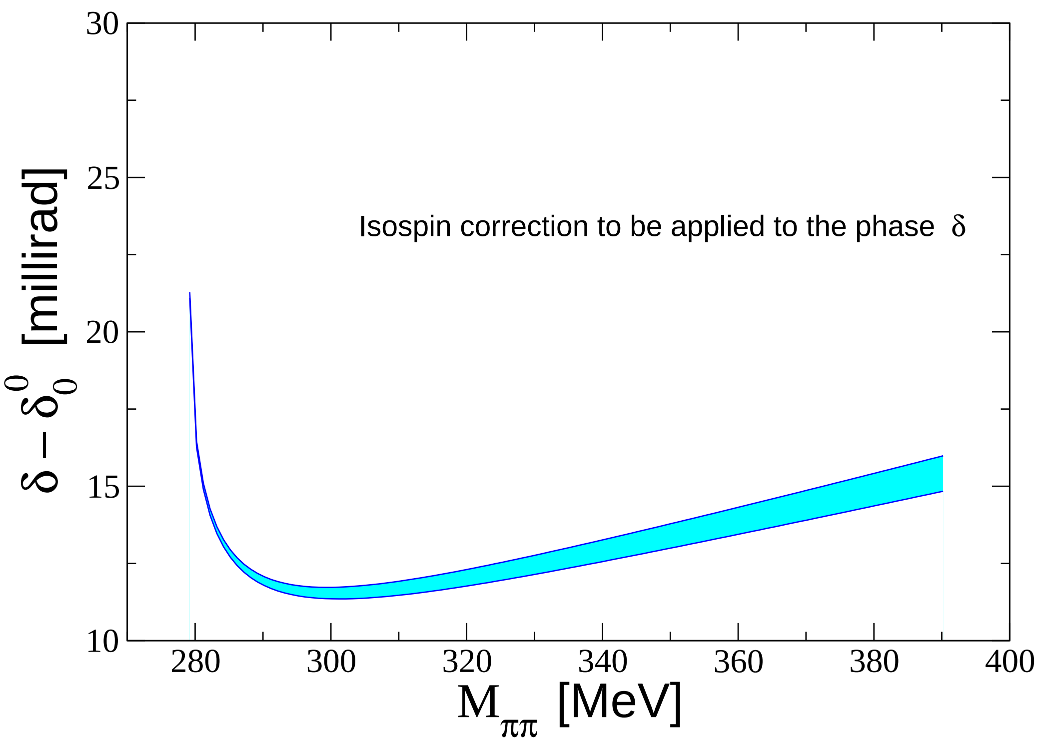
$$\delta_c = \frac{1}{32\pi F_0^2}\left\{4(M_\pi^2 - M_{\pi^0}^2 + s)\sigma_c(s) + (s - M_{\pi^0}^2)\left(1 + \frac{3}{2R}\right)\sigma_0(s)\right\}$$

$$R = \frac{m_s - \frac{1}{2}(m_d + m_u)}{m_d - m_u} \simeq 37 \pm 4 \quad \text{(preliminary)}$$

also: A. Nehme, PRD 69 (2004) 094012; EPJC 40 (2005) 367; V. Cuplov and A. Nehme, hep-ph/0311274

S. Descotes-Genon and M. Knecht, in progress

Including isospin-breaking correction



$$a_0 = 0.233 \pm 0.016\ (stat) \pm 0.007\ (syst)$$

$$a_2 = -0.0471 \pm 0.0011\ (stat) \pm 0.0004\ (syst)$$

No constraints, NA48/2 coll., EPJC 54 (2008) 411

Downward shift $\delta a_0 \simeq -0.02$ due to isospin-breaking correction!

Conclusions

- Cusp in the $K^+ \to \pi^+\pi^0\pi^0$ invariant mass distribution (also, in $K_L \to 3\pi^0$, $\eta \to 3\pi^0$):
  - *Emerges in the kinematically allowed region*
  - *Allows extracting the values of* a_0, a_2
- Cusp in the phase of the formfactor in K_{e_4} decays:
  - $\delta_c \neq 0$ *at charged pion threshold, Watson theorem modified*
  - *Isospin-breaking effects crucial for confronting theory with experiment*
- Non-relativistic effective theories:
  - *The most efficient tool to parameterize amplitudes in the cusp region*
  - *Electromagnetic effects can be systematically included*